INTER-UNIVERSAL TEICHMÜLLER THEORY: A PROGRESS REPORT

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"Travel and Lectures"

§1. Comparison w/Earlier "Teich. Theories"

Classical Complex Teich. Theory:

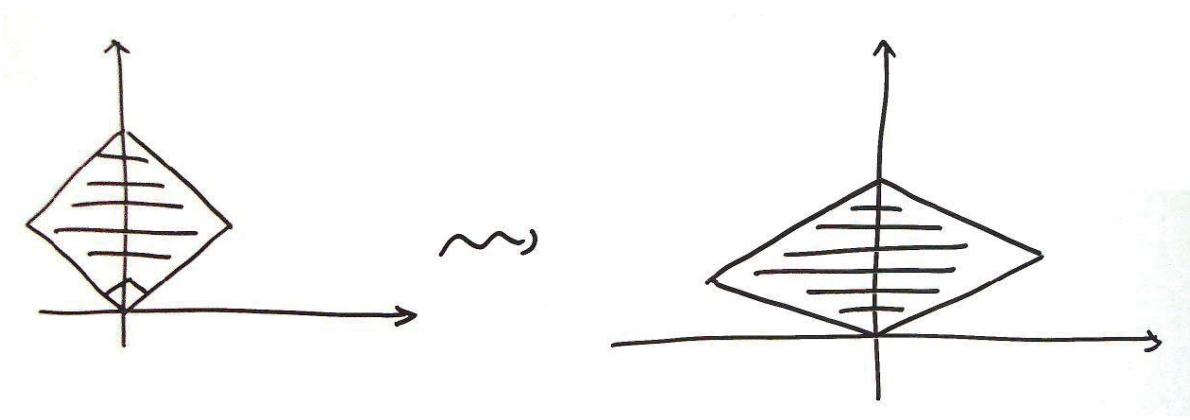
Relative to canonical coord. $z = x + iy$ (assoc'd to a square diff.) on the Riemann surface, Teichmüller deformations given by

$$z \mapsto \zeta = \xi + i\eta = Kx + iy$$

— where $1 < K < \infty$ is the dilation factor.

Key point: one holomorphic dimension, but two underlying real dimensions,

of which one is dilated, while the other is held fixed!



p-adic Teich. Theory:

· p-adic canon. liftings of a hyp. curve in pos. char. equipped with a nilp. ind. bun.

· Frobenius liftings over ord. locus of moduli stack, tautological curve — cf. Poincaré upper half-plane, Weil-Petersson metric/$\mathbb{C}$.

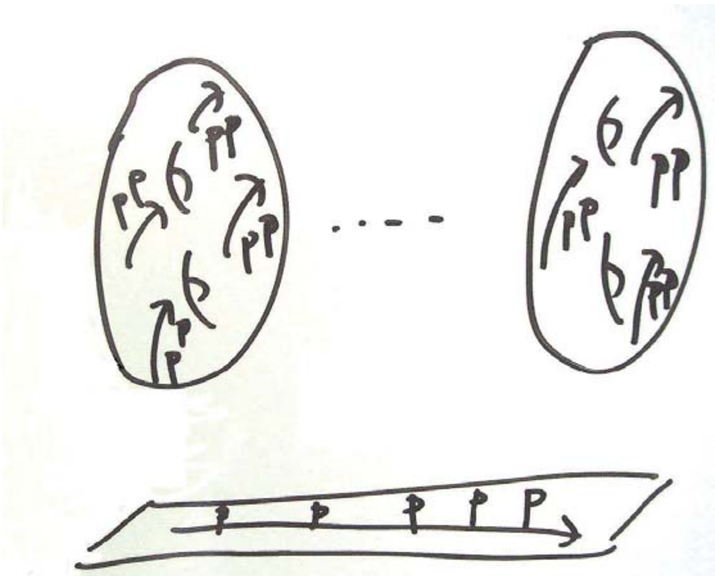
Analogy between IUTeich and pTeich:

scheme theory $\longleftrightarrow$ scheme theory/$\mathbb{F}_p$

"log" no. field $\longleftrightarrow$ pos. char. hyp. curve

once-punct'd ell. curve/NF $\longleftrightarrow$ nilp. IB

log-Θ-lattice $\longleftrightarrow$ p-adic can. + Frob. lift.



§2. Two Underlying Dims. of Arith. Fields

Addition and Multiplication, Cohom. Dim.:

Regard ring structure of rings such as $\mathbb{Z}$ as

one-dim. "arith. hol. str."!

— which has

two underlying comb. dims.!

$$(\mathbb{Z}, +) \quad \curvearrowleft \quad (\mathbb{Z}, \times)$$

1-comb. dim.   1-comb. dim.

— cf. two coh. dims. of abs. Gal. gp. of

- (totally imag.) no. field $F/\mathbb{Q} < \infty$
- p-adic local field $k/\mathbb{Q}_p < \infty$

as well as two underlying real dims. of

- $\mathbb{C}^\times$

Units and Value Group:

In case of p-adic local field $k/\mathbb{Q}_p < \infty$, one may also think of these two underlying comb. dims. as follows:

$$\mathcal{O}_k^\times \subseteq k^\times \twoheadrightarrow k^\times/\mathcal{O}_k^\times \ (\cong \mathbb{Z})$$

1-comb. dim.   1-comb. dim.

— cf. complex case: $\mathbb{C}^\times = \mathbb{S}^1 \times \mathbb{R}_{>0}$

In IUTeich, we shall

deform the hol. str. of the NF

by

dilating the val. gps. via the theta fn.

while

keeping the units undilated

§3. <u>The Log-Theta-Lattice</u>

<u>Noncomm. (!) Diagram of Hodge Theaters</u>:

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \\
\dots & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \dots \\
 & & \uparrow & & \uparrow & & \\
\dots & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \dots \\
 & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & &
\end{array}$$

<u>Analogy between IUTeich and pTeich</u>:

each "HT" $\bullet \longleftrightarrow$ scheme theory/$\mathbb{F}_p$

$\Big\uparrow = \mathfrak{log}$-link $\longleftrightarrow$ Frob. in pos. char.

$\longrightarrow = \Theta$-link $\longleftrightarrow$ $\left(p^n/p^{n+1} \rightsquigarrow p^{n+1}/p^{n+2}\right)$

Thus, 2-dims. of diagram
$\longleftrightarrow$ 2-cb. dims. of p-adic loc. fld.

$\mathfrak{log}$-Link:

At nonarch. v of NF F, ring strs. on either side of $\mathfrak{log}$-link related by non-ring. hom.

$$\log_v : \overline{k}^\times \to \overline{k}$$

— where $\overline{k}$ is an alg. cl. of $k \stackrel{\text{def}}{=} F_v$.

Key point: $\mathfrak{log}$-link is compatible with isom.

$$\Pi_v \stackrel{\sim}{\to} \Pi_v$$

of arith. fund. gps. Π_v on either side, with natural actions via $\Pi_v \twoheadrightarrow G_v \stackrel{\text{def}}{=} \text{Gal}(\overline{k}/k)$; also, compatible with global Galois gps.

At arch. v of F, $\exists$ an analogous theory

Θ-Link:

At bad nonarch. v of NF F, ring strs. on either side of Θ-link related by non-ring. hom.

$$\mathcal{O}^{\times}_{\overline{k}} \ \overset{\sim}{\to} \ \mathcal{O}^{\times}_{\overline{k}}$$

$$\Theta|_{l\text{-tors}} = \left\{ q^{j^2} \right\}_{j=1,\dots,(l-1)/2} \ \mapsto \ q$$

— where $\overline{k}$ is an alg. cl. of $k \overset{\text{def}}{=} F_v$.

Key point: Θ-link is compatible with isom.

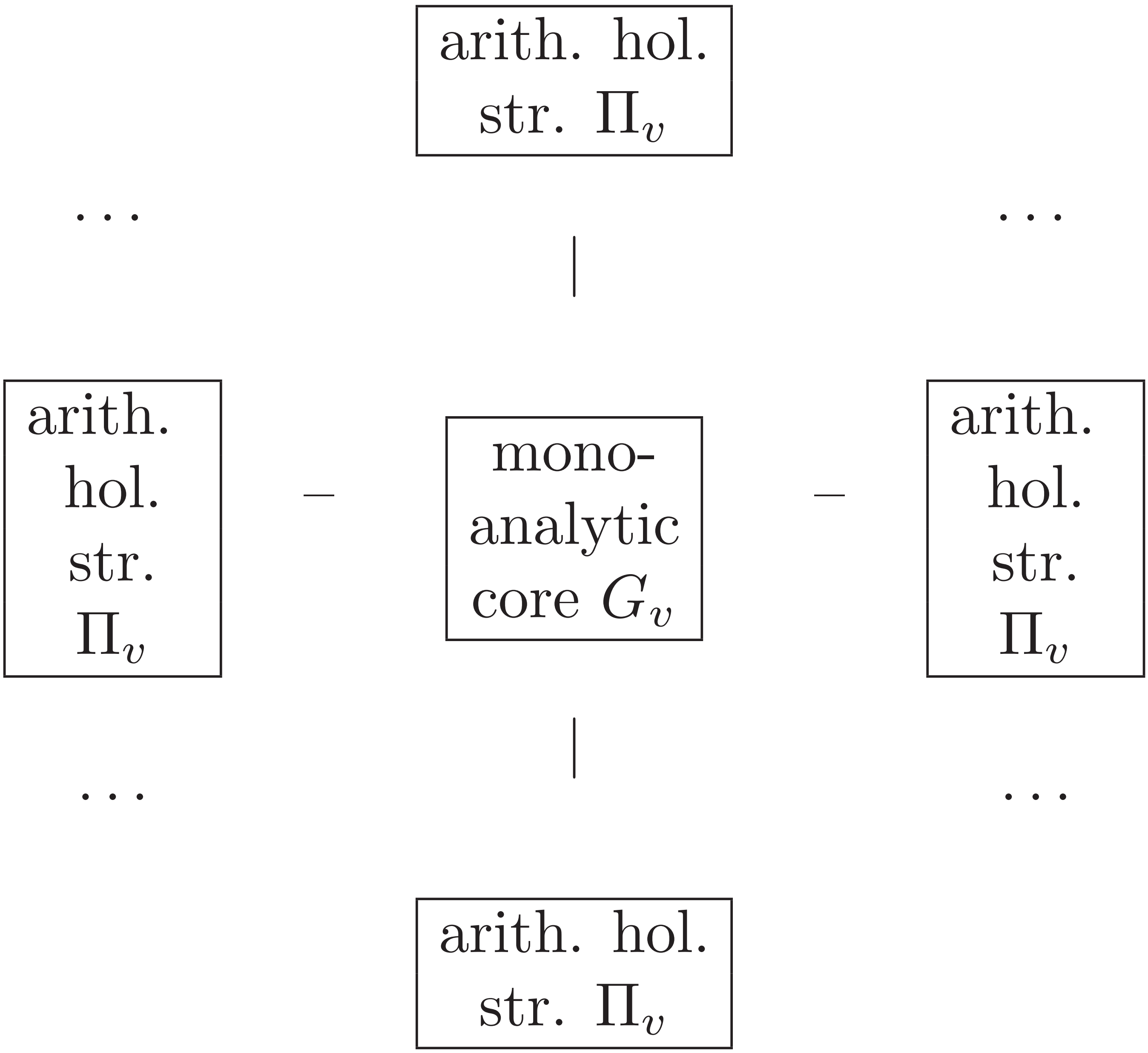
$$G_v \ \overset{\sim}{\to} \ G_v$$

— where $G_v \overset{\text{def}}{=} \text{Gal}(\overline{k}/k)$ — and natural actions on $\mathcal{O}^{\times}_{\overline{k}}$.

At good nonarch./arch. v of F, define analogously, using product formula.

Note: ring str. rigid wrt/$\mathfrak{log}$-link [cf. Π_v!],
but not wrt/Θ-link [cf. G_v! $\widehat{\mathbb{Z}}^\times \curvearrowright \mathcal{O}^\times_{\overline{k}}$!]

Note: "Galois portion" of log-Θ-lattice
$\rightsquigarrow$ étale-picture — cf. cartes. vs. polar coords. for Gaussian int. $\int_0^\infty e^{-x^2} dx$:



§4. Inter-universality and Anab. Geom.

Note that $\mathfrak{log}$-link, Θ-link [i.e., Θ-dilation!] incompatible with ring strs.:

$$\log_v : \overline{k}^{\times} \to \overline{k}$$

$$\Theta|_{l\text{-tors}} = \left\{ q^{j^2} \right\}_{j=1,\ldots,(l-1)/2} \mapsto q$$

— hence with basepoints arising from

- · scheme-theoretic pts., i.e., ring homs.!
- · Gal. gps. regarded as field str. automs.!

Consequence: As one crosses $\mathfrak{log}$-, Θ-links, one only knows "Π_v", "G_v" as abstract top. gps.! Thus, can only relate the bps., "universes", ring/scheme theory in domain, codomain of $\mathfrak{log}$-, Θ-links by applying

anabelian geometry!

§5. Expected Main Results (work in progress!!)

Apply theory/ideas of tempered anab. geo., Étale Theta Fn., Frobenioids, and Topics in Abs. Anab. Geo. III to conclude:

Expected Main Theorem: One can give an explicit, algorithmic description, up to mild indeterminacies, of the left-hand side of the Θ-link — i.e., of "$\Theta|_{l\text{-tors}}$" — relative to the [a priori, "alien"!] ring str. on the right-hand side of the Θ-link.

Key point: coric nature of $G_v \curvearrowright \mathcal{O}^{\times}_{\overline{k}}$!

— cf. analogy with Gaussian integral: i.e.,

dfn. of Θ-link, log-Θ-latt. $\longleftrightarrow$ cart. crds.

algo. desc. via anab. geo. $\longleftrightarrow$ pol. crds.

By performing a volume computation concerning the output of the algorithms of the Expected Main Theorem, one obtains:

Expected Corollary: Inequality of Szpiro ($\iff$ ABC) Conjecture.

... cf.

· "Hasse invariant $= d$(Frob. lift.)" in pTeich

· Gauss-Bonnet on a Riemann surface S

$$- \int_S (\text{Poincaré metric}) \ = \ 4\pi(1-g)$$